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Waves on the erodible bed of an open channel

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In the first part of this work the stability of the erodible bed of a stream with a free surface is studied within the framework of classical hydraulics, in which the velocity variation with depth is reduced to a single mean velocity and the bed friction is related in a general way to the local depth and mean velocity. Only two-dimensional motions can be studied in this way. In considering bed friction and a difference in phase between deposition and the mean velocity gradient along the channel, this work combines aspects of earlier studies of Exner and Kennedy.

In the absence of a phase difference between erosion and mean velocity, the analysis proceeds without linearization. When the development from an equilibrium flow down a uniform slope is considered, the bed waves formed under subcritical flows are found to move downstream, while those under supercritical flows upstream; in both régimes bed waves are damped. In each case the side of a wave facing in the direction of motion is the steeper.

When a phase shift is introduced as well, the analysis is carried forward only after linearization. The primary effect of bed slope and friction is a shift in the ranges of phase angle for which growth can take place and a corresponding alteration in the wavelengths for maximum growth. Friction also reduces bed-wave celerity. Consideration is given to the physical processes represented by the artifice of a phase difference between erosion and mean velocity gradient.

The second section of this investigation concerns two-dimensional potential flow over a wavy stream bed. This problem is considered from a point of view different from that adopted previously by Kennedy; a modified criterion is proposed for the maximum Froude number at which bed waves will form. It is in better agreement with measured data.

In the third part of this paper, the potential analysis is extended to include a class of three-dimensional motions. The conditions are found for the formation of dunes (bed waves 180° out of phase with the surface waves above) and antidunes (the two in phase). The criterion separating dunes and antidunes for two dimensions is found to give a lower limit on the Froude number for antidunes of the more general three-dimensional class. In the antidune régime the streamwise perturbation to the velocity can change sign between surface and stream bed; the limit for this is determined.

The erosion equation relating changes in bed level to local stream speed is generalized to include sediment convection in two dimensions. With this equation, three distinct régimes of bed-wave motion are found: at low Froude numbers, dunes moving downstream; at higher Froude numbers, antidunes moving upstream; and, finally, antidunes moving downstream. The criterion separating the last two régimes is just that mentioned above as the limit of flows whose velocity perturbation has the same sign from surface to stream bed. It is argued that this fundamental change in the flow marks also the end of the region of growth of small bed waves. It is found that three-dimensional dunes and upstream-moving antidunes can exist beyond the two-dimensional limit, but that the latter applies for all bed waves such that the ratio of stream depth to wavelength (d/λ) is small. This explains why the modified criterion for two dimensions provides for small d/λ an envelope for data obtained from observations of a wide variety of bed forms, but fails to do so for larger d/λ. The celerity of bed waves beneath three-dimensional flows is discussed. It is suggested that a class of waves occurring in natural streams will move more slowly than would two-dimensional waves in similar conditions.

The work concludes with a comparison of several methods of modelling erosive flows.

1. Introduction

Without resorting to empiricism, we cannot at present do more than predict the order of magnitude of the sediment moved downstream by a natural river or experimental channel. Even with the full use of correlations of data describing many streams, gross errors are often made in estimating the mean charge of sediment, undoubtedly as a result of the wide variations in the form of natural streams and in the material making up their beds (the two are of course related). The bed waves, sand bands, and meanders found in an erodible stream bed depend for their existence and motion along the channel on local changes in the velocity and depth and on corresponding adjustments in the load of sediment carried with the stream. Since adequate means have not been developed to predict even the mean sediment flux, we cannot hope for a quantitatively accurate description of phenomena dependent on small variations in sediment charge. What can be hoped for is an elucidation of the mechanical principles governing the interaction of a stream with its erodible boundaries, a qualitative description of the major features of bed waves, and a suggestion of the measurements which should be made and of the ways in which they may usefully be correlated.

The mathematical analysis of the coupled waves on an erodible stream bed and in the water flowing above has followed two paths. In the earliest studies, the fluid motion was described using one-dimensional hydraulics; more recently, potential motions have been studied. The two ways of modelling an erodible stream differ mainly in their descriptions of the motion of the water. In all the work known to me, the erosive processes have been characterized by a relationship of the form

$$\partial\zeta/\partial t \propto \partial u/\partial x$$

relating the rate of deposit ($\partial\zeta/\partial t$) to the velocity gradient ($\partial u/\partial x$) and hence to the variation in the sediment-carrying ability of the flow In the potential model the velocity of interest is that at the stream bed. To account for a gradual adjustment of sediment load after a local change in the flow, the velocity gradient may be taken to be that some distance upstream of the point at which the deposition is calculated.

Although the hydraulic and potential models can be examined using very simple mathematical techniques, their properties have not been fully set forth in the literature. In the present work, the combined effects of friction and a difference in phase between deposition and velocity gradient will be considered using the hydraulic model, and three-dimensional potential flows will be examined. The interaction between the stream and its bed will be described as in the past, although a generalization is necessary in dealing with three-dimensional motions of the fluid. It is to be expected that these models will have certain characteristics common to all erosive streams, but not that their detail will be duplicated in real streams.

2. Two-dimensional waves: hydraulics

The flows in natural streams are not two-dimensional, even setting aside the inevitable turbulent fluctuations. Nevertheless, for mathematical convenience, it has usually been assumed in the analysis of river flows that the velocity vector is confined to a plane or even, following classical hydraulics, that the motion is essentially unidirectional. Initially, we shall make the same assumption, partly because of the intrinsic interest and tractability of the two-dimensional case, partly to establish standards of comparison for the three-dimensional motions to be examined later.

In the first part of this section (§ 2 (*a*)) we shall study perturbations to an equilibrium flow down a uniform slope, assuming the mean fluid motion to change so slowly (in time) that it can always be described with sufficient accuracy by steady-state momentum and continuity equations. The alteration of the effective fluid density by the transported material will also be neglected, and the charge of sediment will be taken to be dependent solely on the local mean stream velocity. Exner has studied the development of bed waves beneath a frictional flow (see Leliavsky 1959) under broadly similar restrictions. However, as Leliavsky reports it, Exner's analysis is inconsistent in retaining the unsteady acceleration term of the momentum equation while rejecting from the continuity equation the term representing depth variation in time. His analysis is thus made more difficult without consistently attaining a greater generality.

In § 2 (*b*) we shall generalize Exner's model by relating the deposit of sediment to the velocity some distance upstream. In introducing a lag between velocity variations and the erosive processes we follow Kennedy (1963).

(*a*) *The influence of bed slope and friction*

Since we shall assume for the time being that the erosion or deposition of bed material at any point is dependent on the local mean velocity, these results are applicable, if at all, to slowly changing flows in which the sediment load is always able to adopt the values appropriate to local conditions. On the positive side, the equations governing this restricted development are so simple that they combine to form a unidirectional wave equation giving explicit results without linearization or restriction to sinusoidal perturbations.

The equation relating changes in the bed level $\zeta(x,t)$ and variations in the charge of convected bed material is

$$\partial\zeta/\partial t + \partial L/\partial x = 0, \tag{1}$$

where x is distance measured in the direction of the stream, t is time, and L is the effective volume flux of moving bed material per unit width of the channel. The word 'effective' is introduced because voids are left in the sediment when it is deposited on the stream bed.

If we assume that the charge of sediment is dependent on the local mean velocity only, i.e. that $L = L(u)$, we may write

$$\partial\zeta/\partial t + m\,\partial u/\partial x = 0 \quad \text{with} \quad m = dL/du, \tag{2}$$

where m, the slope of the sediment charge *vs.* velocity curve, is a length characterizing the mobility of the bed material. We expect that $m > 0$ in all real situations.

The continuity and momentum equations to be associated with this erosion equation are

$$uh = q, \tag{3}$$

and

$$u\frac{\partial u}{\partial x} + g\left(\frac{\partial \zeta}{\partial x} + \frac{\partial h}{\partial x}\right) + \beta\frac{u^2}{h} = 0, \tag{4}$$

where $\beta > 0$ is a non-dimensional friction coefficient, q is the volume flow rate per unit width of channel, and h is the water depth. On combination, equations (3) and (4) give

$$\frac{\partial u}{\partial x} = \frac{u}{h(1-F^2)}\frac{\partial \zeta}{\partial x} + \frac{\beta}{g}\frac{u^3}{h^2(1-F^2)}, \tag{5}$$

with $F^2 = u^2/gh = u^3/gq$, F being the Froude number based on the local velocity and depth.

We use equation (5) relating quasi-steady bed forms and velocity distributions to cast the erosion equation (2) into

$$\frac{\partial \zeta}{\partial t} + \frac{mu}{h(1-F^2)}\frac{\partial \zeta}{\partial x} + \frac{m\beta}{g}\frac{u^3}{h^2(1-F^2)} = 0. \tag{6}$$

A motion consistent with the assumptions made above and of practical interest is the spontaneous development from an initially uniform flow down a uniform slope—the 'normal' flow on that slope, in the terminology of hydraulics. To investigate this situation we take

$$\zeta = -\alpha x + \zeta'(x,t),$$

where $\alpha > 0$ is the initial uniform bed slope and ζ' is the developing perturbation of the bed. From equation (4) we find that the parameters of the basic flow are related by

$$F_0^2 = u_0^2/gh_0 = \alpha/\beta,$$

where F_0 is the Froude number based on mean flow parameters. We have here identified β with the friction coefficient of the fundamental flow; this is equivalent to assuming β to be independent of the changes occurring in the perturbed flow.

Using these relationships, we can write equation (5) as

$$\frac{\partial u}{\partial x} = \frac{u}{h(1-F^2)}\frac{\partial \zeta'}{\partial x} + \beta\frac{u}{h}\left(\frac{F^2-F_0^2}{1-F^2}\right). \tag{7}$$

The last term of this equation combines the contradictory influences of bed slope and friction. A question of particular interest is the position of the turning points of the velocity distribution relative to maxima of ζ', the perturbation to the bed shape. This relationship indicates immediately whether the crest will be eroded by a locally accelerating flow or augmented by a decelerating one (for $m > 0$). The two cases to be considered are dealt with in table 1. In both cases, erosion may be expected at the crests of bed waves, deposition in the troughs.

Case	At crest	For $\partial\zeta'/\partial x = 0$
$F, F_0 < 1$	$F > F_0$	$\partial u/\partial x > 0$
$F, F_0 > 1$	$F < F_0$	$\partial u/\partial x < 0$

TABLE 1. Velocity gradients at crests of bed waves.

We turn next to a rewritten form of the erosion equation (6),

$$D\zeta'/Dt = \partial\zeta'/\partial t + c\,\partial\zeta'/\partial x = \epsilon, \tag{8}$$

giving the change of a perturbation travelling along the bed with celerity

$$c = mu/h(1-F^2). \tag{9}$$

Note that $c \gtrless 0$, for $F \lessgtr 1$, giving the downstream and upstream motions of bed waves traditionally (but not necessarily correctly) associated with subcritical and supercritical flows. The rate of change is

$$\epsilon = \frac{\beta m u}{h}\left(\frac{F_0^2-F^2}{1-F^2}\right) < 0 \quad \text{for both} \quad F \lessgtr 1.$$

This last result is consistent with the comments based on equation (7). In the absence of friction, we should have a neutrally stable propagating wave; the damping of the bed waves may be attributed to the combined action of friction and slope in displacing the turning points of the velocity distributions.

Finally, we calculate

$$\frac{dc}{du} = \frac{m}{h}\left[\frac{2+F^2}{(1-F^2)^2}\right] > 0 \quad \text{for both} \quad F \lessgtr 1.$$

Examined with equation (7) in mind, this result suggests that in general the steeper faces of bed perturbations are those facing in the direction of motion. This prediction is in accord with observation; see for example the plates of Kennedy's (1963) paper.

(b) *The influence of a phase difference between erosion and velocity gradient*

The results of the preceding investigation of the stability of a stream bed are negative, in the sense that perturbations are found to be damped in all realistic situations. Since bed waves do in fact appear spontaneously, some vital physical process must have been absent from the model used. To obtain a mathematical model in which growth is possible, we use the artifice introduced by Kennedy (1963) (and suggested earlier by Cartwright 1959)—an arbitrary phase difference between erosion and mean velocity. Consideration will be given later to the physical basis of such a phase difference. To introduce it without complicating the analysis beyond utility, we shall linearize the equations by retaining only the terms of the lowest orders in the perturbations to the mean velocity and depth.

While the erosion equation (2) will be formally retained as the basis of the analysis, it will henceforth represent a wide range of interactions between the stream and its bed material. In essence, the assumption we shall make about the erosive processes is that the rate of deposit ($\partial\zeta/\partial t$) is proportional to some physical quantity (perhaps the bed shear stress) whose magnitude is proportional to that of the velocity perturbation, but whose relative phase has not been specified.

The linearized form of equation (7) is

$$(1-F_0^2)\frac{h_0}{u_0}\frac{\partial u'}{\partial x} = \frac{\partial \zeta'}{\partial x} + 3\beta F_0^2 \frac{u'}{u_0},$$

where we have set $u = u_0 + u'$. Using the relationship $F_0^2 = \alpha/\beta$, we obtain

$$m\,\partial u'/\partial x = c_0(\partial\zeta'/\partial x + 3\alpha u'/u_0), \tag{10}$$

where $c_0 = mu_0/h_0(1-F_0^2)$ (cf. equation (9)).

With a spatial lag δ between erosion and velocity, we have

$$[\partial L/\partial x]_x = m[\partial u'/\partial x]_{x-\delta},$$

linking the velocity to the variation in the load of convected bed material. The symbolism $[\;]_x$ means 'to be evaluated at x'. Equations (1) and (10) give

$$[\partial\zeta'/\partial t]_x + c_0[\partial\zeta'/\partial x + 3\alpha u'/u_0]_{x-\delta} = 0 \tag{11}$$

(cf. equation (8)).

We consider sinusoidal perturbations,

$$\zeta' = a\,e^{ik(x-ct)} \quad \text{and} \quad u' = bu_0 e^{ik(x-ct)},$$

where $c = c_r + ic_i$ is the complex bed-wave celerity; b is in general a complex quantity also, but a is a small real positive quantity. Substitution of these perturbations in equation (10) gives

$$a/b = mu_0/c_0 + 3i\alpha/k, \tag{12}$$

and in equation (11)

$$c = c_0 e^{-i\theta}(1 + 3i\alpha c_0/mu_0 k)^{-1}, \tag{13}$$

when use is made of the relationship (12). Here $\theta = k\delta$ is the angular phase lag.

For compactness we introduce

$$f = 3\alpha c_0/mu_0 k = 3\alpha/h_0(1-F_0^2)\,k, \tag{14}$$

a parameter expressing the importance of bed slope and friction. For negligible friction, $f \to 0$; while $f \to \pm\infty$ for situations in which slope and friction are of great importance. Also, $f \gtrless 0$ for $F \lessgtr 1$.

The expression (13) for the complex celerity may now be written as

$$c = c_0 e^{-i\theta}/(1+if) = c_0(1+f^2)^{-\frac{1}{2}}\,e^{-i(\theta+\phi)} \quad \text{with} \quad \phi = \tan^{-1}f. \tag{15}$$

Friction reduces the magnitude and alters the phase of the complex celerity. The condition for neutral stability is $\sin(\theta+\phi) = 0$, and that for stationary (i.e. non-translating) bed waves is $\cos(\theta+\phi) = 0$. Figure 1 shows the dependence on f of the ranges of θ for various modes of bed-wave development.

We seek now the conditions for the most rapid initial growth of perturbations, in the expectation that the corresponding waves are those which will in practice appear on the bed of a channel. Thus we anticipate that the physical processes giving rise to bed-wave growth are not associated with specific angular phase differences, but do perhaps give rise to a fixed spatial lag. This point will be taken up later.

We have

$$\zeta' = a\,e^{ik(x-ct)} = a\,e^{kc_i t}e^{ik(x-c_r t)}.$$

Then $A = a e^{kc_i t}$ is the amplitude of the simple travelling wave, and

$$[dA/dt]_0 = akc_i$$

is the initial growth rate of the moving train of perturbations. Hence the condition for maximum initial growth is

$$\partial[\partial A/\partial t]_0/\partial k = a\mathscr{I}\{\partial(kc)/\partial k\} = 0.$$

Using equations (14) and (15), we find it to be

$$\tan\theta = -\frac{2f^3+(1+f^2)\,\theta}{1+3f^2-f(1+f^2)\,\theta}\quad \begin{cases} \to -\theta & \text{as} \quad f\to 0, \\ \to 2/\theta & \text{as} \quad |f| \to \infty. \end{cases} \tag{16}$$

The latter limiting result is satisfied by $\theta = 49{\cdot}2^\circ$, $196{\cdot}2^\circ$, and other values lying only a little above $\theta = N\pi$, $N = 2, 3, \ldots$. Only the second of these roots falls within the regions of positive growth shown in figure 1; it is the limiting value for both $f \to \pm\infty$, $F_0 \lessgtr 1$.

For the limiting frictionless case ($f \to 0$) the first two roots of equation (16) are $\theta = 116{\cdot}2^\circ$ and $281{\cdot}5^\circ$; their relevance to the growth of bed waves may be seen in figure 1. The intermediate conditions for maximum initial growth are shown there, too; each branch stays within a region of the graph associated with one mode of bed development. Thus we see that friction does not alter the kind of bed wave which grows most rapidly at a particular Froude number, although the wavelength for maximum growth is changed. The movement of bed waves is predicted to be downstream for both the dunes† of subcritical flow and the antidunes† of the supercritical case.

We have followed Kennedy in confining our attention to the first two possible maxima of the growth rate, one for subcritical, the other for supercritical flow. It is difficult to justify this restriction, especially when it is observed that the growth rate is higher at subsequent maxima.

Comparing the importance of friction and phase shift in this analysis, we note that any lag between erosion and velocity gradient, no matter how small, completely changes the character of the stability problem from one in which all bed waves are damped to one in which waves of certain lengths can grow. Further, the prediction of the direction of motion of antidunes is altered by the introduction of a lag.

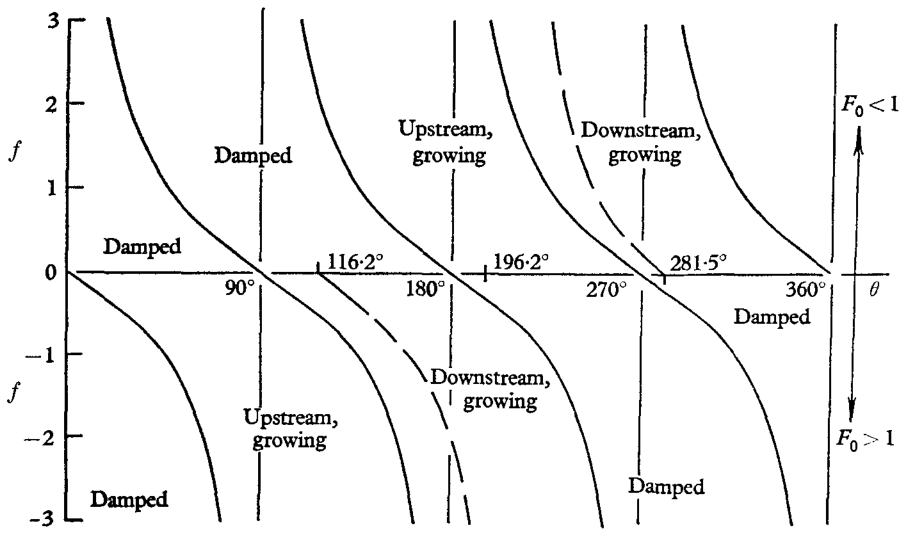

FIGURE 1. Dependence of mode of bed-wave development on friction parameter f, Froude number F_0, and phase angle θ. The full curves represent boundaries between regions of one mode of development. The dashed curves represent conditions of maximum initial growth rate and are both asymptotic to $\theta = 196{\cdot}2.°$

Friction has no such drastic effect. On examining the form of the parameter f (equation (14)), noting that in practical cases $\alpha \sim O(10^{-3})$ while $kh_0 \sim O(1)$, we conclude that only near the critical condition ($F_0 = 1$) can f take on any but very small values. Table 2 represents the values of the friction parameter for several of the flows examined by Simons, Richardson & Albertson (1961) which are nearly critical. The role of friction is greater in subcritical cases, particularly in Run 27. For that case, figure 1 suggests an alteration in the phase angle for maximum growth of $\Delta\theta = 15°$.

† It is difficult to give a comprehensive definition of these terms. Traditionally, the words 'dunes' and 'antidunes' have been used to distinguish between downstream- and upstream-moving bed waves. However, Kennedy found it expedient, in describing the results of his potential analysis, to define antidunes as bed waves in phase with the surface profile and, conversely, dunes as 180° out of phase with the surface. In a frictional flow there is no longer a rigid relationship between the phases of bed and surface and his clear-cut distinction is not possible.

We now turn to the physical basis for the phase difference or spatial lag between velocity and erosion. Two interpretations will be examined: first, a phase difference between the velocity gradient along the channel ($\partial u/\partial x$ of equation (2)) and the bed shear stress which has often been thought of as governing the pick-up of material from the stream bed; secondly, a spatial lag between a local change in the flow and the adjustment of the sediment load to the modified conditions. The implications of the two proposals differ profoundly. The first mechanism would give a specific angular phase shift, independent of the wavelength of the disturbances on the stream bed. This phase difference depends on

Run no.	F_0	f
27	0·92	0·40
30	0·84	0·18
31	1·13	−0·05
39	1·12	−0·05

TABLE 2. Values of f from data of Simons *et al.* (1961)

the fluid motion; a good approximation could be found by considering a rigid wavy bed. The bed material is vital to the second mechanism. Its role is most easily visualized in terms of the phenomenon of saltation, in which a particle once dislodged from the stream bed leaps clear and is carried a discrete distance downstream before again touching the bed. This and more complex processes produce a spatial lag dependent on the bed material and the basic flow, but only weakly influenced by the spacing of the bed waves. Hence the wavelength (or wavelengths) appearing will be that (those) for which the fixed spatial lag corresponds to an angular phase difference favourable to rapid growth. Only if the mechanism is of this nature can any physical significance be ascribed to the conditions of maximum growth rate studied earlier (equation (16)).

Let us next examine the implications of the hypothesis that the most rapid erosion will take place where bed shear is highest, concentrating first on subcritical flows. The crudest concept of the generation of shear stress at the stream bed would have the stress largest where an inviscid theory predicted the highest velocity, that is, at the crests of the bed waves for a subcritical hydraulic flow. If this were the case, a phase difference of 90° would pertain between shear stress and the velocity gradient $\partial u/\partial x$. Benjamin (1959) has improved upon this picture by examining viscous shearing flows over wavy boundaries; since he considered a semi-infinite volume of fluid, his results are directly comparable only to the limiting subcritical case. For several flows at high Reynolds numbers he found the maximum shear stress to occur 30° and 60° upstream of the crests. These results suggest phase differences of 60° and 30° respectively. Referring to figure 1, we see that these values of θ (i.e. 90°, 60°, and 30°) lie in the range of damped waves for subcritical flow. Alternatively, it may be argued quite generally that large shear stresses will occur somewhere on the crests of bed waves in subcritical flows, so that these waves would inevitably be eroded away if the local shear stress were the vital factor in determining the rate of erosion. We conclude that the mechanism studied by Benjamin does not give rise to phase shifts conducive to bed-wave development in subcritical flows, although it is doubtless relevant to the generation of surface waves by wind, the process in which he was directly interested.

If we look at supercritical flows, we find a quite different situation, for the highest velocities (and presumably shear stresses) occur somewhere in the troughs of bed waves. Again a phase shift of 90° is appropriate in the simplest inviscid model. Under these circumstances, the mechanism under consideration could give rise to wave growth and, as can be seen from figure 1, the waves for $F_0 > 1$, $\theta \simeq 90°$ move upstream. Viewed in this way, the hydraulic model indicates the upstream-moving waves which have been observed under supercritical flows.

Finally, we conclude that neither of the two mechanisms proposed to justify a phase shift or spatial lag can alone account for all the bed forms which occur in nature.

3. Two-dimensional waves: potential flow

Kennedy (1963) has made an exhaustive study of this problem, using a velocity potential to specify two-dimensional flow over long-crested bed waves. Unfortunately, one fundamental result of his analysis is incorrect—the criterion giving the maximum Froude number for the formation of bed waves with a particular ratio of depth to wavelength. The re-examination necessary to revise this criterion will provide an introduction to the three-dimensional cases to be examined later.

The reader should note that the notation used subsequently differs in a few respects from that used until now. The symbols F, c_0, and u are used slightly differently; the symbol β has an entirely new meaning; and the mean depth is identified by a new symbol, d. These changes are made either to maintain established conventions or to facilitate comparisons with the work of Kennedy.

(*a*) *The motion of the fluid*

Kennedy's first step is a re-examination of the classical potential flow satisfying appropriate conditions at a free surface and a kinematic restriction at a sinusoidally perturbed stream bed. He starts with a stationary wave pattern in water of uniform depth D and mean stream speed U, as shown in figure 2, noting that streamlines other than the surface (near $y = 0$) and level bottom ($y = -D$) can be chosen to represent various wavy stream beds. The streamline oscillating about d_1, such that $0 < d_1 < D$, corresponds generally to a perturbed supercritical flow beneath which antidunes (bed waves in phase with the surface disturbance) are established. A streamline near d_2, such that $0 < D < d_2$, corresponds to a subcritical basic motion under which dunes (180° out of phase with the surface) occur on the bed.

Kennedy points out that the class of motions considered must obey Airy's condition

$$kU^2/g = \tanh(kD),$$

relating the parameters of small free oscillations in water of constant depth ($k = 2\pi/\lambda$ is the wave-number). The wave-numbers consistent with this condition lie in the range $0 < k < g/U^2$, and the corresponding wavelengths are such that $\infty > \lambda > 2\pi U^2/g$.

Kennedy argues further that the minimum wavelength in the water sets a limit on the occurrence of bed waves. This criterion may be expressed as a maximum Froude number at which bed waves are possible, as follows:

$$F^2 = \frac{U^2}{gd} = \frac{1}{kd}\left(\frac{U^2k}{g}\right)_{\max.} = \frac{1}{kd}. \tag{17}$$

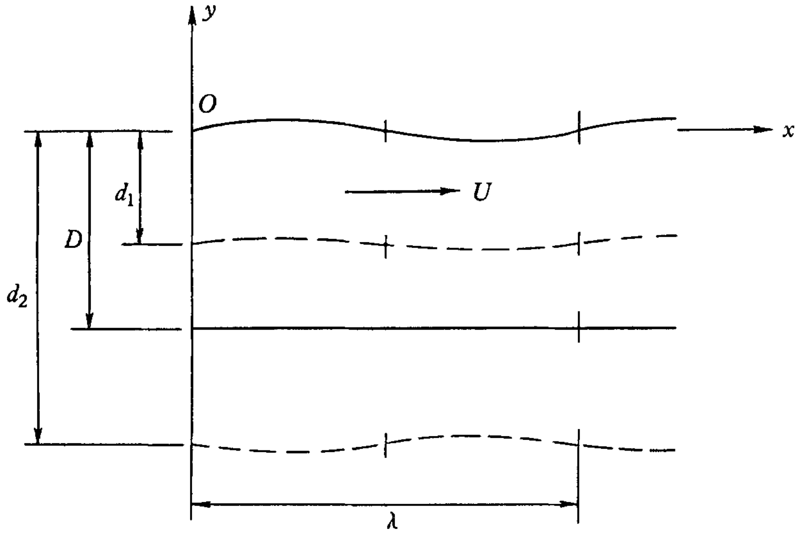

FIGURE 2. Symbols in two-dimensional potential analysis.

Three objections to this approach and its conclusion can be advanced. It seems unnecessary to tie the analysis of a wavy bed to that for a level one, or, looked at another way, it is not obvious that all forced motions of the liquid are contained in the class of free motions. We may argue too that very short bed waves do commonly form in rapid natural streams. Finally, as Kennedy points out himself, for high values of F and low values of kd the observations lie consistently beyond the limit given above.

The velocity potential representing a steady sinusoidal perturbation on a uniform stream and satisfying the dynamic condition at the free surface is

$$\phi = Ux + UA[e^{ky} + \{(1-G_0^2)/(1+G_0^2)\}\, e^{-ky}] \cos(kx),$$

with $G_0^2 = kU^2/g$, where A is a real constant. No restriction has been made on the nature of the bottom; it may be either level or wavy. We now take it to be the latter and to be given by

$$y = -d + a\sin(kx).$$

Then the ratio of surface and bed-wave amplitudes is

$$R = [\cosh(kd) - \sinh(kd)/G_0^2]^{-1}.$$

$R > 0$ for $G_0^2 > \tanh(kd)$, antidunes; $R < 0$ for $G_0^2 < \tanh(kd)$, dunes. No restriction on G_0 is to be inferred from this approach (save that it be positive).

In addition to the bed waves for $G_0 < 1$ admitted by Kennedy, which include both dunes and antidunes, and the singular case of free oscillations above a level bed, we have now admitted bed waves for which $G_0 > 1$, invariably antidunes. This added class consists of short bed waves producing little surface disturbance. (Note that $R \to 0$ as $k \to \infty$, $\lambda \to 0$, for any water depth d.) These short antidunes should not be confused with the dune ripples recorded by Simons *et al.* (1961) and by other observers.

(*b*) *Criterion for formation of bed waves*

It may be argued that Kennedy's condition $F^2 = 1/kd$ prescribes with reasonable accuracy the range of observable bed waves. Having found it to be groundless, we are now faced with the task of establishing another criterion of nearly similar form to replace it. To do this, we study the motion of the bed waves themselves. This requires a consideration, in general terms at least, of the processes of erosion and deposition of the bed. The criterion derived is thus essentially different from Kennedy's. His was obtained from a consideration of the dynamics of the fluid alone, and hence denied that certain flows were possible even over rigid sinusoidally fluted plates. The condition replacing it is linked to the process of erosion and states only that certain lengths of bed waves will not appear spontaneously on erodible stream beds. Expressed in mathematical terms, this new condition is the limit of instability of the bed to erosive processes. We need not carry out the stability analysis explicitly here. Kennedy's discussion of the ranges of stability can be used as the basis of this revision of his work.

We may use the earlier steady-state analysis (that of § 3 (*a*)) on the assumption that the bed waves move very slowly. Introducing into equation (2) the perturbation

$$\zeta = a\,e^{ik(x-c_0 t)}$$

and the corresponding

$$u' = -UAk[e^{-kd} + \{(1-G_0^2)/(1+G_0^2)\}\,e^{kd}]\,e^{ik(x-c_0 t)},$$

we obtain

$$\frac{c_0}{U} = -mk\,\frac{1-G_0^2\tanh(kd)}{G_0^2-\tanh(kd)}, \tag{18}$$

a result given by Kennedy in another form.

For the cases he considered, $G_0 < 1$, so that

$$c_0/U < 0 \quad \text{for} \quad G_0^2 > \tanh(kd), \text{ antidunes;}$$
$$c_0/U > 0 \quad \text{for} \quad G_0^2 < \tanh(kd), \text{ dunes.}$$

If the restriction on G_0 is relaxed, a new limit is set for upstream propagation:

$$G_0^2 = \coth(kd),$$

or

$$F^2 = \frac{U^2}{gd} = \frac{1}{kd}G^2_{\text{max.}} = \frac{\coth(kd)}{kd}. \tag{19}$$

The significance of this criterion may be seen as follows. The ratio of velocity perturbations at the bed and at the surface is found to be

$$u'(-d)/u'(0) = \cosh(kd) - G_0^2\sinh(kd).$$

The perturbation changes sign between surface and bed when $G_0^2 > \coth(kd)$. It is this fundamental change in the flow pattern which gives rise to another régime of downstream propagation of bed waves. We argue further that it marks also the end of the region of instability of bed waves of small amplitude.

In figure 3 the two criteria ((17) and (19)) are plotted with experimental results taken from Kennedy's paper. The region shown is that in which they differ significantly.

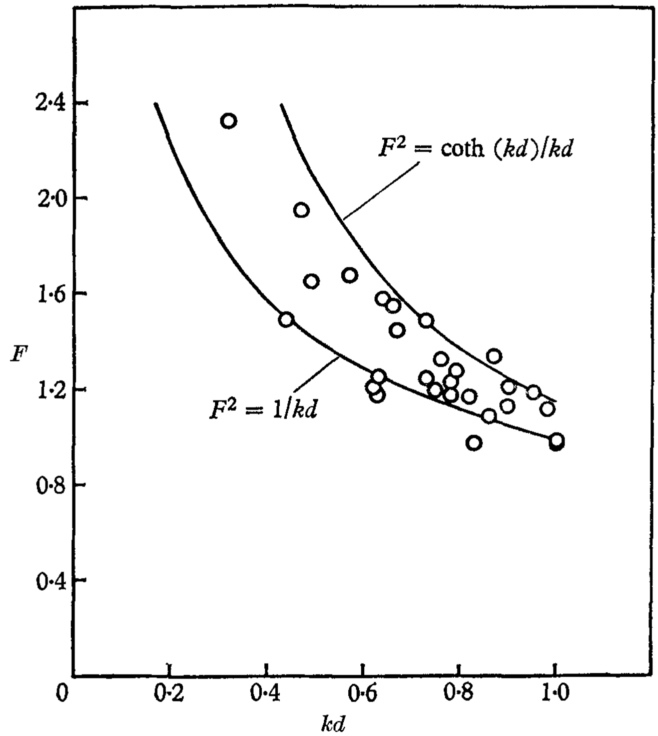

FIGURE 3. Comparison with measurement of criteria for the limit of growing bed waves. The points represent conditions under which bed waves have been seen to form naturally.

Even shorter bed waves are consistent with the motion of the liquid. It is possible that the erosion gives rise to a further region of instability for much smaller wavelengths. On considering the data collected by Kennedy, one may then ask why so few observations do lie well beyond the limiting values proposed here and in Kennedy's work. The answer is very likely that investigators were interested in dunes and banks rather than mere sand ripples, and therefore consistently failed to record the latter phenomena when they did occur. For example, in figure 8 of Simons *et al.* (1961) small ripples are visible on the surface of large banks.

4. Three-dimensional waves

The necessity of studying this more general class of motions becomes apparent on considering the bed forms occurring in nature and in experiments. The motion within a river meandering in an erodible flood plain is obviously three-dimensional. But sharply curving channel banks are not necessary for complex bed configurations. For example, the photographs of Kennedy (1963) and of Simons *et al.* (1961) show three-dimensional bed waves in channels with rigid, non-erodible walls. Exner's studies (reported by Leliavsky 1959) of the bottom of the River Mur reveal a succession of sand banks, lying alternately near the two sides of a slowly curving stream, their positions having no apparent relation to the basic curves. Similar observations are reported by Leopold & Wolman (1957) for the Valley and Brandywine Creeks in Pennsylvania.

For three-dimensional motions, the classical techniques of hydraulics fail. Since only the approach through potential flow is available, friction cannot be taken into account. In most respects the following analysis parallels that of §3, the boundary conditions being again fully linearized. The erosive processes will be supposed to occur wholly on a nearly level stream bed; in effect, the stream will be retained between rigid vertical walls. While the widely meandering river is undoubtedly of ultimate interest in the study of streams with erodible beds, we can now model only flows in nearly straight channels.

(*a*) *The motion of the fluid*

We shall first examine a steady motion over a wavy stream bed, without reference to the much slower motion of the bed waves beneath. The basis of the study is the velocity potential.

$$\phi = Ux + UA[e^{\beta y} + \{(1-G^2)/(1+G^2)\}\, e^{-\beta y}] \cos(kx)\cos(lz), \tag{20}$$

where $\beta = (k^2+l^2)^{\frac{1}{2}}$ and $G^2 = k^2U^2/g\beta$, satisfying the dynamic condition at the free surface. The relationship of this motion to actual flows will be discussed shortly.

If the slightly wavy bottom is given by

$$y = -d + a\sin(kx)\cos(lz),$$

the ratio of the surface-wave amplitude to bed-wave amplitude is

$$R = [\cosh(\beta d) - \sinh(\beta d)/G^2]^{-1}.$$

Then
$$R > 0 \quad \text{for} \quad G^2 > \tanh(\beta d), \text{ antidunes;}$$
$$R < 0 \quad \text{for} \quad G^2 < \tanh(\beta d), \text{ dunes;}$$

or
$$F^2 \gtrless \frac{\beta^2}{k^2}\frac{\tanh(\beta d)}{\beta d} \quad \text{for} \quad \begin{Bmatrix}\text{antidunes,}\\ \text{dunes.}\end{Bmatrix} \tag{21}$$

In figure 4 the critical curves ($R = 0$) separating dunes and antidunes are shown for two cases, the two-dimensional ($\beta = k$) and a case in which three-dimensional waves occur ($\beta = 2k$). We see that the former curve provides an inner limit for the formation of antidunes of any planform, although three-dimensional dunes can occur beyond this limit, as well as antidunes. Kennedy's data (1963, figure 9; and figure 3 of this paper) are in general accord with this theoretical pattern: a few points representing antidunes fall a little below the curve $R = 0$ for $\beta = k$, while a larger number of points representing dunes lie above.

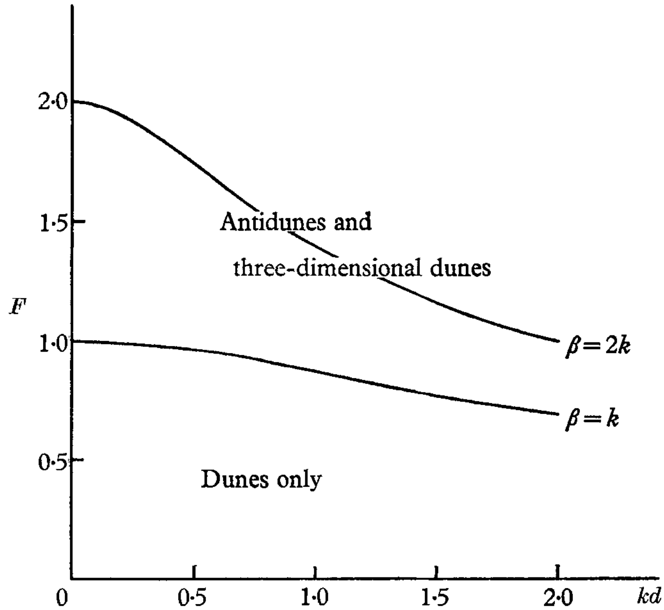

FIGURE 4. Criteria separating dunes and antidunes for two-dimensional waves ($\beta = k$) and for a class of three-dimensional waves ($\beta = 2k$).

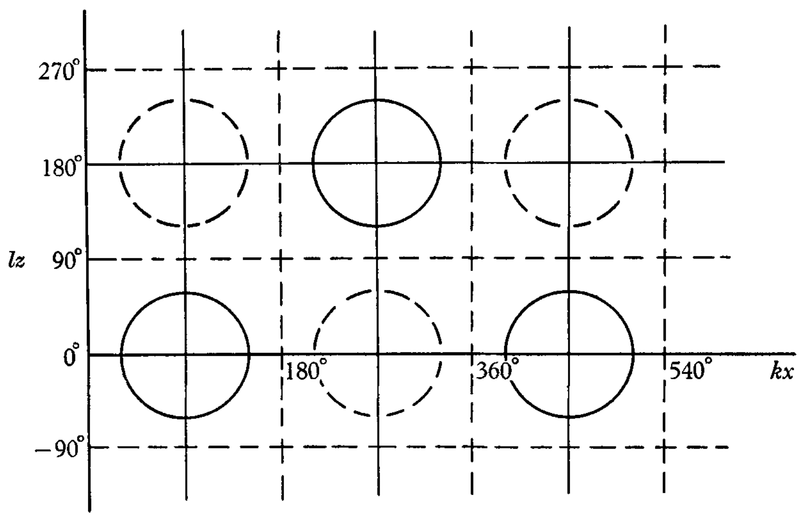

FIGURE 5. Contours of stream bed associated with three-dimensional flow. Dashed lines mark average depth d. Solid lines mark maximum and minimum depths. Dashed curves mark depth $d + \frac{1}{2}a$ (deeps). Solid curves mark depth $d - \frac{1}{2}a$ (banks).

The contours of the stream bed specified above (equation (20)) are sketched in figure 5. We now ask what kind of channel is modelled by this pattern. For the rigid vertical walls of experimental flumes, the condition $w = \partial\phi/\partial z = 0$ must apparently be satisfied at the walls. Then for the simple pattern of sand banks alternating from side to side along the channel, the walls must lie at $lz = 0°$, $180°$. Some doubt is cast on the appropriateness of this choice for natural streams, if we compare the resulting pattern of bed waves with the observations of Exner. Many of the sand banks he studied on the bottom of the Mur were not in contact with the river banks, but lay well out in the channel, as in the mathematical pattern obtained by choosing the lines $lz = -90°, 270°$ for the stream boundaries. This latter choice is open to the objection that $w \neq 0$ on the supposed river banks. We conclude that neither choice describes Exner's observations fully. However, the appropriate identification of channel width with wave-number must lie in the range $b = \pi/l$ to $2\pi/l$.

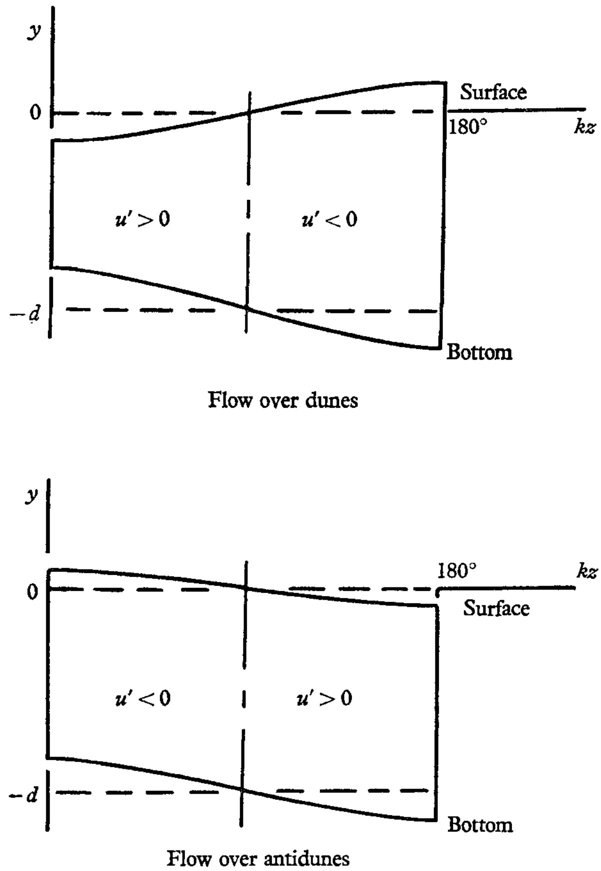

FIGURE 6. Transverse profiles of stream bed and surface.

The variation of the streamwise velocity perturbation with depth can be studied as it was for two dimensions. We find the perturbation to be of constant sign from surface to bed only so long as $G^2 < \coth(\beta d)$. This criterion is satisfied by all dunes (for which $G^2 < \tanh(\beta d)$), and by all upstream-moving antidunes, as has been shown for two-dimensional motions and will later be shown for three dimensions.

The cross-stream profiles of bed and surface are sketched in figure 6 for the cases of dunes and antidunes. The corresponding velocity changes in the two halves of the channel are also indicated. Note too that

$$\Delta Q = aU(R + 1/R - 2)\sin(kx)\cos(lz)$$

gives (to the first order) the change in the volume flux in the x-direction across a vertical line through the flow from surface to bed, while

$$R+1/R-2 \gtrless 0 \quad \text{for} \quad R \gtrless 0,$$

so that the larger flow is in both cases through the half of the channel in which the velocities are lower. This fundamental fact about the lateral displacement of the perturbed stream may also be demonstrated from the variation of w along the centre-line of the stream ($lz = 90°$) or by noting the direction of the streamline curvature and centripetal field required to balance the transverse slope of the surface.

(*b*) *Generalized erosion equation*

We first cast the analysis of sediment convection of § 2 (a) into a form more readily extended to convection in the two dimensions of the horizontal (x, z)-plane. We can take

$$L = nu,$$

with n the effective volume of suspended material per unit area through the flow from surface to bed. The quantity n may be thought of as roughly the thickness of the layer which would be deposited upon complete sedimentation.

For $n = n(u)$ only, we have

$$\partial L/\partial x = (n + u\,dn/du)\,\partial u/\partial x = m\,\partial u/\partial x,$$

as given in equation (2).

In the case of present interest the sediment flux is a two-vector

$$\mathbf{L} = iL_x + kL_z \quad (i,\ k \text{ being unit vectors})$$

and will be assumed in the form

$$\mathbf{L} = n(V)\,\mathbf{u},$$

where $\mathbf{u} = iu + kw$, and $V^2 = u^2 + v^2 + w^2$, with all the velocity components evaluated at the stream bed. This representation for the flux vector does not imply that $v \ll u', w$ at the stream bed. The flux vector represents an integral over a cylindrical section passing vertically through the stream and hence is represented correctly by a horizontal vector even in a three-dimensional flow.

The sediment conservation equation (a generalization of equation (1)) is now

$$\partial\zeta/\partial t + \nabla.\mathbf{L} = 0,$$

or

$$\partial\zeta/\partial t + (dn/dV)\,(\mathbf{u}.\nabla)\,V + n\nabla.\mathbf{u} = 0.$$

For the case of small perturbations from the uniform flow, $u = U$, $v = w = 0$, we have $V^2 \simeq U^2 + 2Uu'$, so that

$$(\mathbf{u}.\nabla)\,V \simeq V\,\partial u'/\partial x.$$

The erosion equation can then be written

$$\partial\zeta/\partial t + m\,\partial u'/\partial x + n\,\partial w/\partial z = 0 \quad \text{(cf. equation (2))},$$

with $m = d(nV)/dV \simeq dL/dV$. These results are correct only to the first order in u'/U.

(c) The celerity of bed waves

We consider a slowly moving pattern given by

$$y = -d+\zeta = -d+a\sin k(x-ct)\cos(lz),$$

and the associated velocity potential, simply that of equation (20) with x replaced by $x-ct$, on the assumption that $c \ll U$. Substituting this potential into the preceding equation, we find that

$$\frac{c}{U} = -\frac{mk^2+nl^2}{\beta}\left[\frac{1-G^2\tanh(\beta d)}{G^2-\tanh(\beta d)}\right].$$

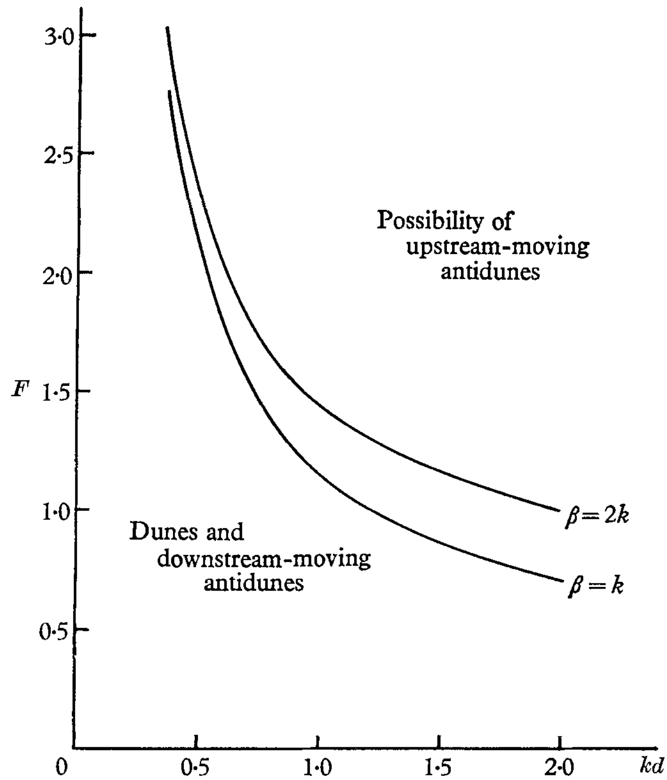

FIGURE 7. Criteria separating different modes of bed development for two-dimensional waves ($\beta = k$) and for a case of three-dimensional waves ($\beta = 2k$).

We expect that $m, n > 0$ in all realistic situations. Then

$$c/U > 0 \quad \text{for} \quad G^2 < \tanh(\beta d), \quad G^2 > \coth(\beta d),$$

$$c/U < 0 \quad \text{for} \quad \coth(\beta d) > G^2 > \tanh(\beta d).$$

The latter case consists solely of antidunes. In the former, the first of the ranges contains only dunes. It has been argued earlier that the second range is one of damping; if this is so, we may expect that no bed waves will form there spontaneously. In figure 7 the criterion

$$G^2 = \coth(\beta d) \quad \text{or} \quad F^2 = \beta^2\coth(\beta d)/k^2\beta d$$

is plotted for two cases, the limiting one of $\beta = k$ (equation (19)), and that for which $\beta = 2k$. Note that the limit found here becomes $F \to 1/kd$ as $kd \to 0$ for all β/k, so that the two-dimensional limit is nearly correct for three-dimensional waves when kd is small. However, for larger values of the depth to wavelength ratio, the criterion separating upstream and downstream-moving antidunes is displaced upwards to higher Froude numbers. We have proposed earlier that this criterion indicates the end of the range of bed-wave growth as well. This

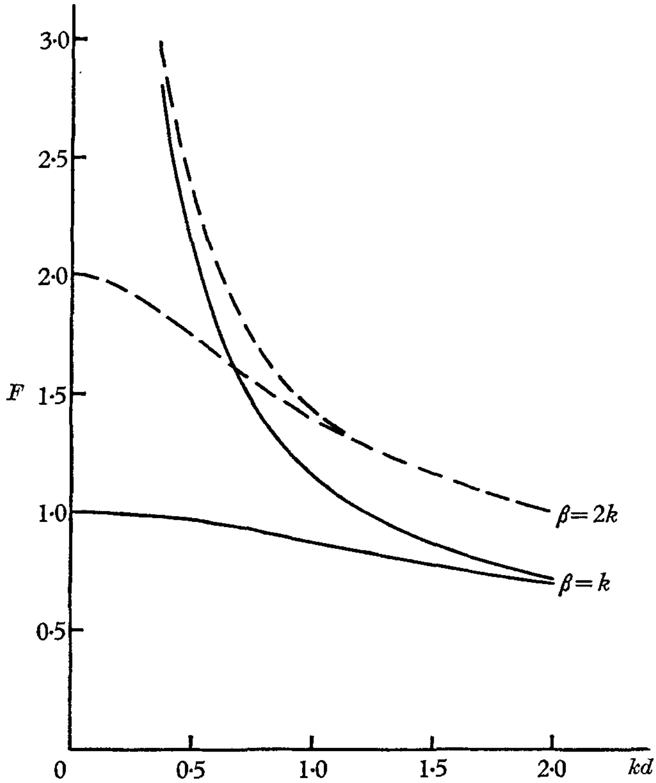

FIGURE 8. Superposition of criteria of figures 4 and 7.

interpretation is consistent with Kennedy's data (1963, figure 9), which lie within the two-dimensional limit for small kd (see figure 3) but elsewhere stray well beyond.

In figure 8 the two sets of criteria (from figures 4 and 7) are shown superposed. The regions contained within the corresponding pairs of limits are those in which upstream-travelling antidunes occur. The pairs of curves converge so rapidly as kd increases that, for all practical purposes, antidunes can exist only when $kd < 2$, in agreement with Kennedy's data.

The pattern of limits revealed by the potential analysis is much more complicated than that suggested by the hydraulic model studied first. The critical condition separating dunes and antidunes has been generalized in two respects, and a second criterion, without a physical basis in a one-dimensional hydraulic flow, has been discovered, with even more profound implications for bed-wave development. These criteria are especially valuable because they do not depend on the values assigned to parameters (such as n and m) necessary to describe the interaction fully. Nor do these criteria depend on the details of the stability analysis or on the nature of the erosive process, although the general nature of the stability considerations must be kept in mind in interpreting the limits in physical terms.

It is of interest to see what values the parameters introduced in this analysis adopt in actual rivers. The somewhat fragmentary data of Leopold & Wolman (1957) and of Exner suggest that the values

$$kd = 0{\cdot}05, \quad F = 0{\cdot}3, \quad l/k = 6$$

are typical of streams in which banks alternate from side to side along straight reaches. We shall now attempt to compare the celerity of the three-dimensional waves formed in these circumstances with that of two-dimensional waves of the same relative length. From equation (18) we have

$$\frac{c}{c_0} = \frac{k}{\beta}\left[1+\frac{n}{m}\left(\frac{l}{k}\right)^2\right]\left[\frac{1-G^2\tanh(\beta d)}{1-G_0^2\tanh(kd)}\right]\left[\frac{G_0^2-\tanh(kd)}{G^2-\tanh(\beta d)}\right].$$

The comparison can now be made without specific knowledge about the bed material, for this influences the ratio c/c_0 only through

$$\frac{n}{m} = \frac{L/V}{dL/dV}.$$

A fair idea of the variation $L(V)$ can be obtained from measurements of mean sediment load. The important features are that $L = 0$ until a critical velocity is attained, and that thereafter the curve $L(V)$ is concave upwards. Then

$$0 < n/m < 1$$

quite generally. Taking $n/m = \frac{1}{3}$, we find $c/c_0 = 0{\cdot}33$ for the conditions given above.

For these conditions, both the hydraulic and potential analyses suggest that bed waves should take the form of dunes moving downstream. This prediction is consistent with the observations of Exner. The alternating banks of the Mur were found to move downstream with a celerity a little above 200 m per year. Leliavsky (1959) mentions other instances of downstream movement of large banks with celerities ranging from 20 to 700 m per year. On the other hand, for the short train of bed waves in the Valley Creek studied by Leopold & Wolman, it seems probable that there is no translation along the channel, for the crest farthest upstream in the reach described is founded on an outcropping of bed rock.

5. Conclusion

Although the preceding results are of some value in themselves, perhaps their greatest utility lies in their implications for further work. The hydraulic model is attractive in its simplicity, and has the advantages that friction can be introduced in a general way, and that rigorous linearization of the boundary conditions is not necessary. However, the role of friction in bed-wave development was not found to be a vital one for the case considered in § 2. Further, the complete neglect of the velocity variation with depth leads to a grossly oversimplified view of the stability of a stream under erosive attack. We must conclude that further analysis of bed waves based on one-dimensional hydraulics is unlikely to be fruitful.

The two-dimensional potential model provides a surprisingly accurate picture of bed-wave growth, as evidenced by figure 3 of this paper and figure 9 of Kennedy's (1963). Some consideration of three-dimensional motions is necessary for a complete interpretation of the two-dimensional results, as has been shown in figures 4 and 7. Nevertheless, further study of two-dimensional motions will probably be valuable. A first step would be to replace the potential motion by a simple shearing flow—rotational, but still inviscid. This would probably give a better estimate of the limits separating the several régimes of the stability problem. But a real improvement in the stability analysis as a whole cannot be made unless the sediment/stream interaction is more realistically described as well.

Three-dimensional investigations are bound up with the important question of sinuosity of rivers. But analytical success seems to dictate strict linearization while real rivers meander widely, and, when even slightly sinuous, exhibit great variations in depth along their courses. We still lack methods of studying these largest erosion waves which mould an entire stream channel.

I should like to express my thanks to Mr S. W. Law, who pointed out to me an important error in the derivation of § 4.

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